Simultaneous Diophantine approximation and linearization of $\mathbb{C}^2$ maps

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Outline of talk

- $\mathbb{C}$ analytic (holomorphic) maps

- Diophantine approximation

- Simultaneous Diophantine approximation

- $\mathbb{C}^2$ analytic maps

General theme - interaction between dynamical systems and number theory

Discrete dynamical systems on $\mathbb{C}$

- $f : \mathbb{C} \to \mathbb{C}$ holomorphic or meromorphic

- Orbit $\{ z_0, z_1 = f(z_0), z_2 = f(z_1), \ldots \}$

- When can we linearize?
  Schröder equation for $f(z) = \lambda z + \mathcal{O}(z^2)$:

$$\sigma \circ f = \lambda \sigma$$

- If this has a solution on some domain D, then $\hat{f} \equiv \sigma^{-1} \circ f \circ \sigma$ is linear.

- This all works if $|\lambda| \neq 0, 1$

If $|\lambda| = 1$, no solution if $\alpha \equiv \log(\lambda)/(2\pi i)$ is rational. If it is irrational, we get *small divisors* in the power series for σ.

Briggs' universal foolproof method for solving hard problems

(Pat. Pend.)

Given: Hard problem H.

1. Find a transformation $\phi : \mathrm{H} \mapsto \mathrm{E}$, where E is an easy problem.
2. Solve E.
3. Take ϕ^{-1} of the solution of E.

Formal power series solutions

Given the series

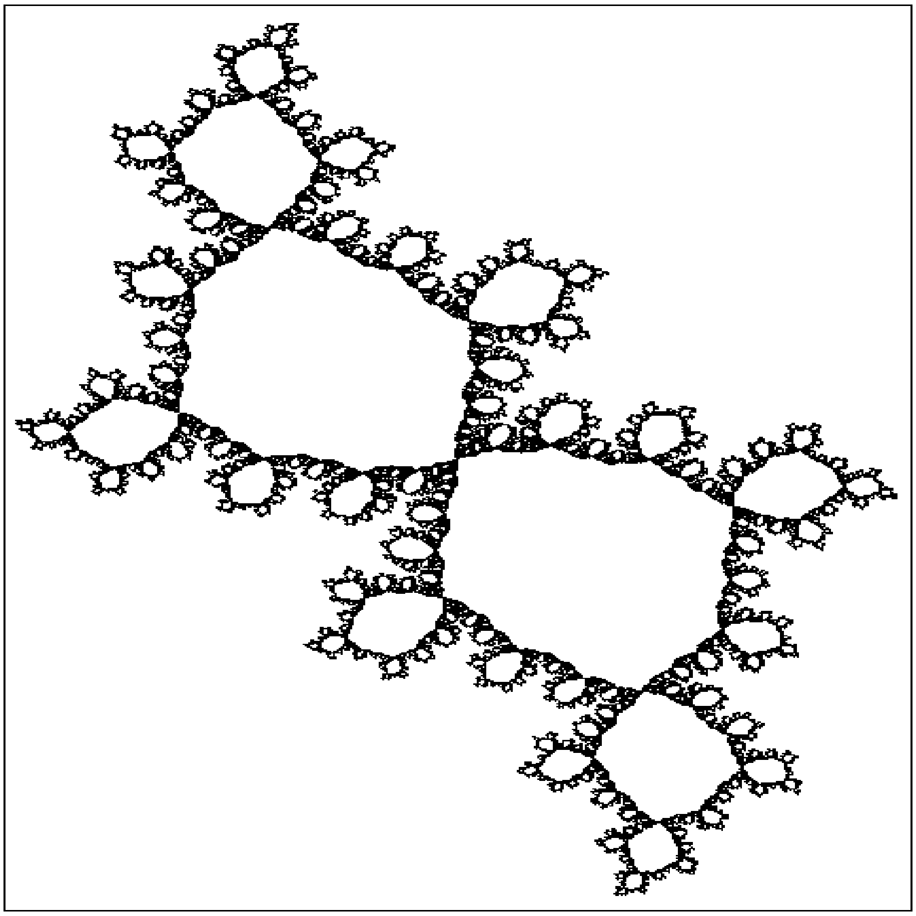
$$f(x) = f_1 x + f_2 x^2 + f_3 x^3 + \dots \qquad (f_i \in \mathbb{C})$$

we want the coefficients $\sigma_1, \sigma_2, \sigma_3, \cdots$ in the expansion

$$\sigma(x) = \sigma_1 x + \sigma_2 x^2 + \sigma_3 x^3 + \cdots$$

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ f_2 & f_1^2 - f_1 & 0 & \cdots \\ f_3 & 3f_1 f_2 & f_1^3 - f_1 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

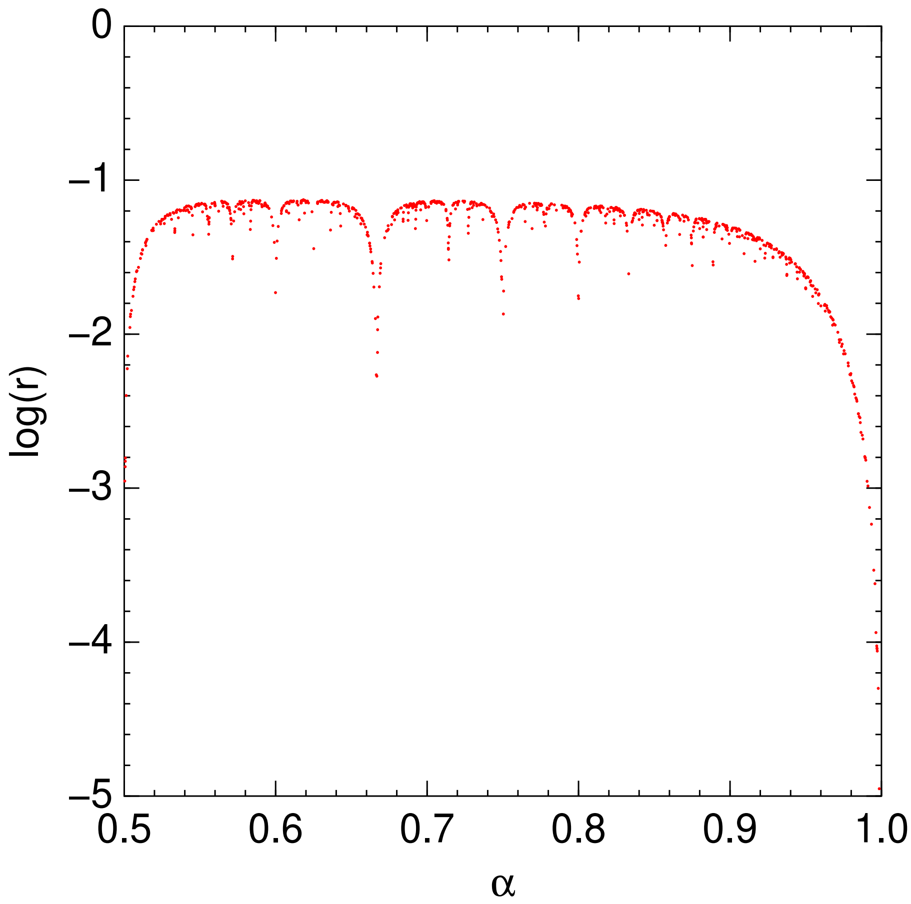
Siegel disks



Julia set J for $e^{2\pi i\alpha}z + z^2$ with $\alpha = \sqrt[3]{1/4}$

$$\begin{array}{ccc} J & \xrightarrow{f} & J \\ \sigma\big\downarrow & & \sigma\big\downarrow \\ B(0,r) & \xrightarrow{f_1\times} & B(0,r) \end{array}$$

The Siegel radius r



The logradius of convergence of the Siegel disk for $\exp{(2\pi i\alpha)}z + z^2$

Diophantine approximation in 1 dim

Goodness of approximation of p/q to $\alpha \notin \mathbb{Q}$:
$c(\alpha, p, q) \equiv q|q\alpha - p|$

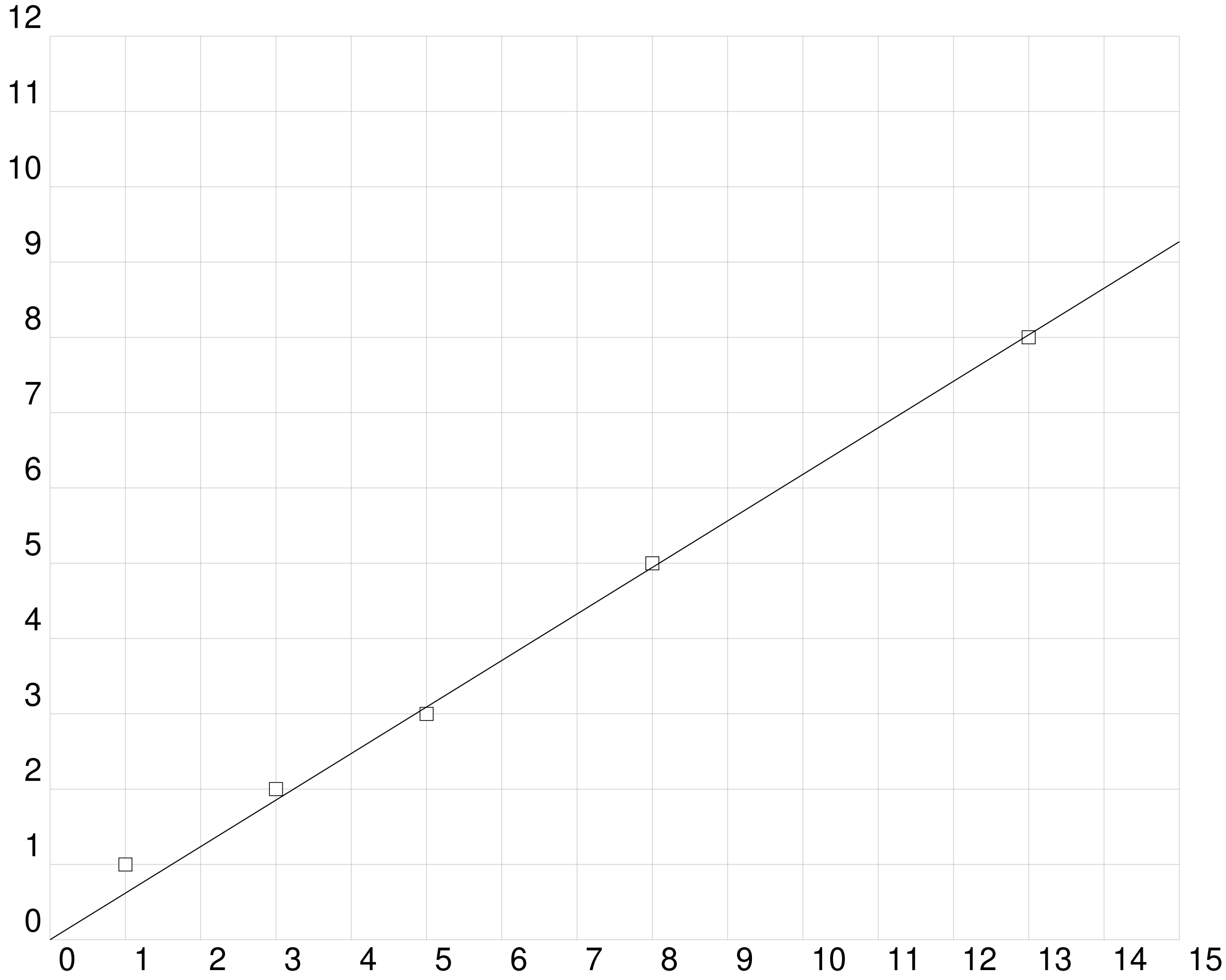
$\exists$ infinitely many rationals p/q such that
$|\alpha - p/q| < 1/q^2 \Rightarrow c(\alpha, p, q) < 1$.

Approximation constant of α is
$c(\alpha) \equiv \liminf_{q \to \infty} \min_p c(\alpha, p, q)$

Note: if $\lambda = \exp(2\pi i\alpha)$, then
$4|\alpha q - p| < |\lambda^q - 1| = 2 \sin \pi|\alpha q - p| \leqslant 2\pi|\alpha q - p|$
where $p = \lfloor \alpha q \rceil$

The one-dimensional diophantine approximation constant is
$c_1 = \sup_{\alpha \in \mathbb{R}} c(\alpha) = 1/5^{1/2}$

$\mathbb{Z}^2$ view of best approximants



Best approximants provide successive minima of vertical distance of line from lattice points.

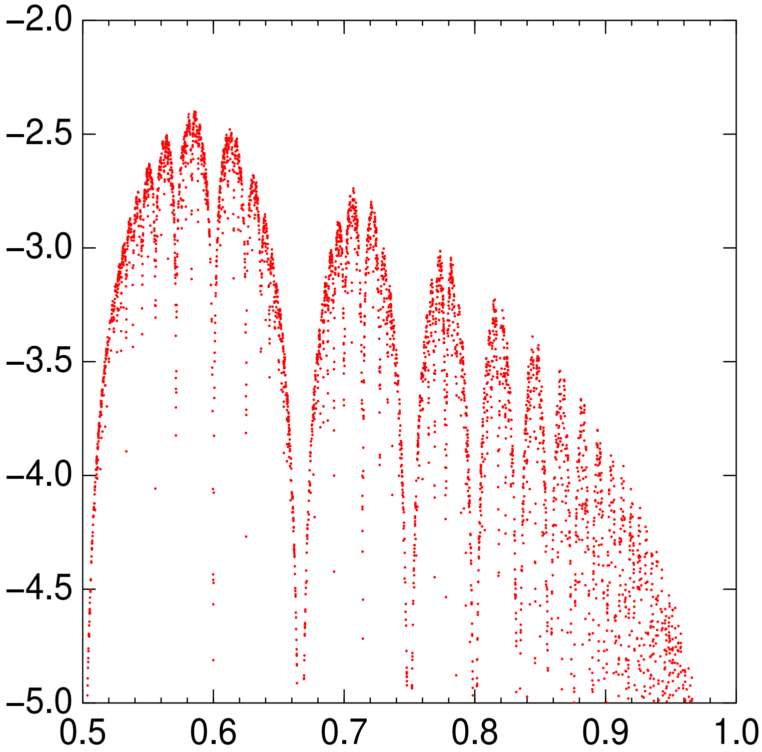
The Bruno* function

Expand an irrational $\alpha = [a_0, a_1, a_2, \ldots]$ in a continued fraction and consider the sequence of convergents (best rational approximants)

$$p_0/q_0, p_1/q_1, p_2/q_2, \ldots.$$

Define

$$B(\alpha) = \sum_{i=0}^{\infty} \log(q_{i+1})/q_i$$



*Бруно

Maximizing bad approximability

'Diophantine' (badly approximable) irrationals α are those with $c(\alpha) > 0$. 'Bruno' irrationals are those with $B(\alpha) < \infty$ (a larger set)

What is 'worst approximable' irrational? We might try to:

1. maximize c

2. minimize growth rate of denominators ($g \equiv \liminf_{n\to\infty} q^{1/n}$)

3. maximize the Bruno function

The first two optima are achieved at $\alpha = (\sqrt{5} - 1)/2$.

The basic theorems

Bruno (1971) proved that $B(\alpha) < \infty$ is a sufficient condition for $\exp(2\pi i\alpha)z + \mathcal{O}(z^2)$ to be linearizable.

Yoccoz (1988,1995) proved that all such f are linearizable iff $B(\alpha) < \infty$.

The key idea: prove that

$$|\log r + B(\alpha)|$$

is bounded, where r is the radius of the Siegel disk

Diophantine approximation in 2 dims

Define
$\|\boldsymbol{\alpha}\|_q \equiv \min_{\mathbf{p}\in\mathbb{Z}^2} \max\left(|q\alpha_1 - p_1|, |q\alpha_2 - p_2|\right)$

For $\mathbf{p} = (p_1, p_2) \in \mathbb{Z}^2$, $q \in \mathbb{Z}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, let

$$\begin{aligned} c(\boldsymbol{\alpha}, q) &= q\|\boldsymbol{\alpha}\|_q^2, \\ c(\boldsymbol{\alpha}) &= \liminf_{q\to\infty} c(\boldsymbol{\alpha}, q). \end{aligned}$$

The two-dimensional sup-norm simultaneous diophantine approximation constant is then

$$c_2 = \sup_{\boldsymbol{\alpha}} c(\boldsymbol{\alpha}) = ?$$

More generally, replace $\|\cdot\|_q$ with any *radius* function ρ, and q by any *height* function h

E.g. $\rho(q, p_1, p_2) = |q + \alpha_1 p_1 + \alpha_2 p_2|$
(approximation by a linear form)

Cubic fields

A *cubic number field* $\mathbb{K}$ is a set

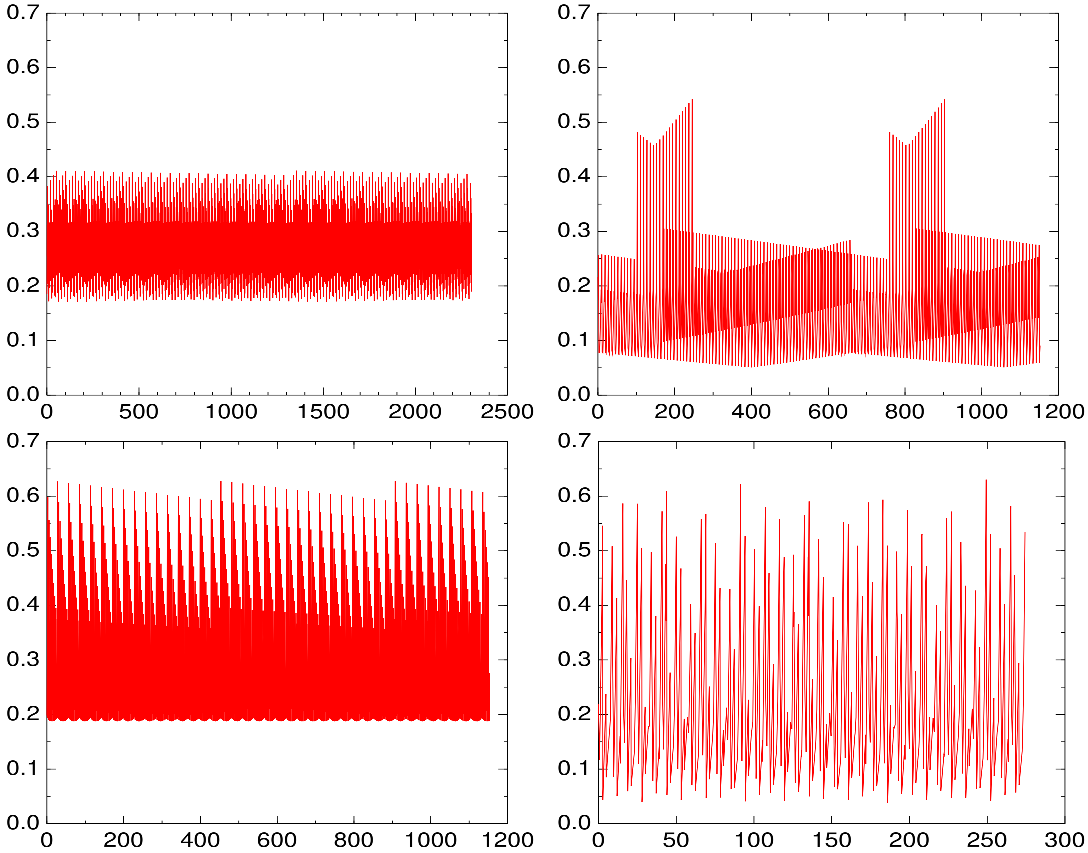
$$\{\, a_0 + a_1\zeta + a_2\zeta^2 \mid a_0, a_1, a_2 \in \mathbb{Q} \,\}$$

where ζ is a root of a monic irreducible cubic polynomial with integer coefficients.

An *integral basis* of $\mathbb{K}$ is a basis for the free $\mathbb{Z}$-module $\mathbb{Z}_{\mathbb{K}}$, the ring of algebraic integers. For all the fields below, $\{1, x, x^2\}$ is an integral basis.

d	polynomial
-23	$x^3 + x^2 - 1$
-31	$x^3 - x^2 - 1$
-44	$x^3 - x^2 - x - 1$
-59	$x^3 + 2x - 1$
$+49$	$x^3 + x^2 - 2x - 1$
$+81$	$x^3 - 3x - 1$

$c(\alpha, q)$ in cubic fields



$c(\alpha, q)$ vs. $\log q$ for $(1, \alpha)$ being integral bases of fields of discriminant $-23, -44, +49, +81$.

Growth rates

For the 2d problem, define growth rate

$$g(\boldsymbol{\alpha}) \equiv \liminf_{n\to\infty} h^{1/n}((q, p_1, p_2)_n)$$

For simultaneous approximation, Lagarias has shown $g > 1.272$ for all pairs. I found the smallest $g = 1.563$ in the field $d = -23$, and an average of 3.07.

For linear forms, I have found an average of $1.72864 \approx \sqrt{3}$.

Almost nothing else is known in this area.

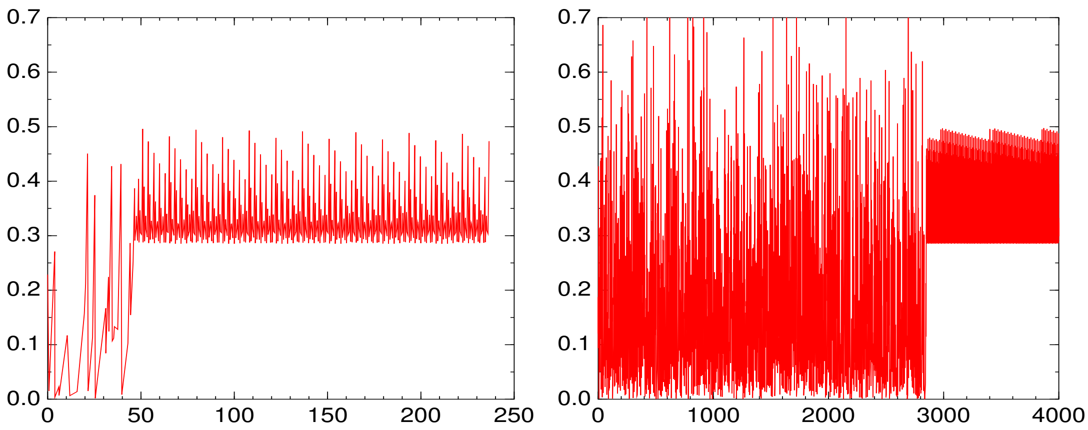
Maximizing bad approximability

What is 'worst approximable' pair of irrationals? We might try to:

1. maximize $c(\boldsymbol{\alpha})$
2. minimize growth rate $g(\boldsymbol{\alpha})$
3. maximize the Bruno function $B(\boldsymbol{\alpha})$

$$\boxed{\text{The only known fact is that } c \text{ is maximized in the field of discriminant } 49 \text{ when restricted to integral bases of cubic fields}}$$

We can approach $c = 2/7$ from below.



Vaughan Clarkson

has recently discovered the first algorithms that for any height and radius find a super-sequence of the best approximants.

$$\rho h(X) \equiv (\rho(X), h(X)), \qquad X = (q, p_1, p_2) \in \mathbb{Z}^3$$

$$h\rho(X) \equiv (h(X), \rho(X))$$

$$\begin{array}{c} L(X,Y,Z) \equiv \\ [\rho h(X) < \rho h(Z) \wedge \rho h(Y) < \rho h(Z)] \wedge h\rho(X) < h\rho(Y) \\ \vee \\ \neg[\rho h(X) < \rho h(Z) \wedge \rho h(Y) < \rho h(Z)] \wedge \rho h(X) < \rho h(Y) \end{array}$$

The algorithms operate by a sequence of changes of basis of the lattice $\mathbb{Z}^3$, at each step 'consulting' L to decide which new basis to take

Exact real arithmetic

has a natural rôle here.

The algorithms depend on the interrelation between rationals (exact in computer representation) and reals (usually not exact).

> We need L to always return a correct result for all inputs, but with floating-point of *any fixed precision*, however large, it will eventually fail

This can be avoided using recent work of Vuillemin, Ménissier-Morain, Potts etc. We use a functional representation of reals with lazy evaluation.

The 2-dimensional Bruno function

Given:

$\lambda_j = \exp(2\pi i \alpha_j) \qquad j = 1, 2$

Define:

$\Omega(m) = \inf_{\substack{2 \leqslant j+k \leqslant m \\ j,k \geqslant 0}} \min\left(\left| \lambda_1^j \lambda_2^k - \lambda_1 \right|, \left| \lambda_1^j \lambda_2^k - \lambda_2 \right| \right)$

Then:

$B(\boldsymbol{\alpha}) = -\sum_{j=0}^{\infty} 2^{-j} \log \Omega(2^{j+1})$

Theorem:

if $B(\boldsymbol{\alpha}) < \infty$, the $\mathbb{C}^2$ map

$$f \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{bmatrix} + \mathcal{O}(z^2)$$

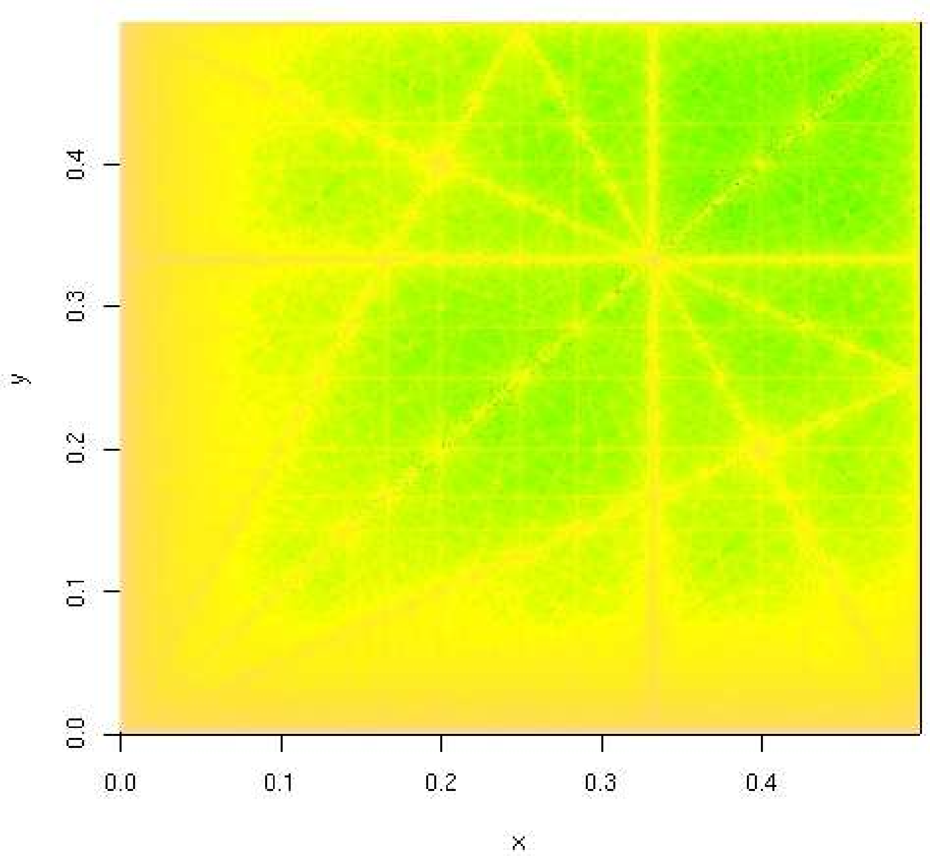
is linearizable. Converse?

Computing the 2-dimensional Bruno function

Clarkson's method allows us to find

$$\Omega(m) = \inf_{2 \leqslant j+k \leqslant m} \min\left(\left|\lambda_1^j \lambda_2^k - \lambda_1\right|, \left|\lambda_1^j \lambda_2^k - \lambda_2\right|\right)$$

Thus finding $B(\boldsymbol{\alpha})$ is now feasible.



Detailed picture:

`www.btexact.com/people/briggsk2/nh500.jpeg`

2 dimensional linearization

We solve for σ in

$$\sigma \circ f = \lambda\sigma$$

as a 2-variable formal power series with $f(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2) + \mathcal{O}(z^2)$.

Let $f = \sum_I f_I z^I$ be any 2-variable power series $(I = (i_1, i_2))$ and D the domain of convergence.

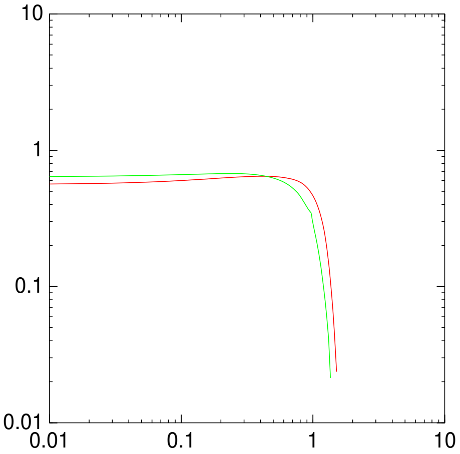
Then D is logconvex, meaning that the set $\log D \equiv \{ (\log|z_1|, \log|z_2|),\ z = (z_1, z_2) \in D\}$ is convex.

$\boxed{\text{Our main question is: what measure of irrationality of } (\alpha_1, \alpha_2) \text{ controls the 'size' of } D?}$

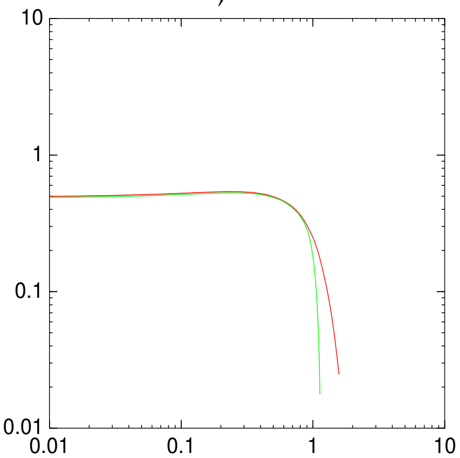
Candidates: Bruno function B, growth rate r, approximation constant c.

Results 1, negative discriminant

We show the logradius of convergence for the two components of σ^{-1} as a function of $(\log|z_1|, \log|z_2|)$

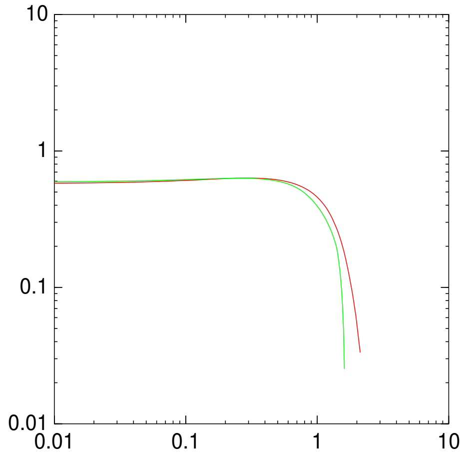


$$d = -23, \ \ B = 2.801, \ \ r = 1.504, \ \ c = 0.449$$

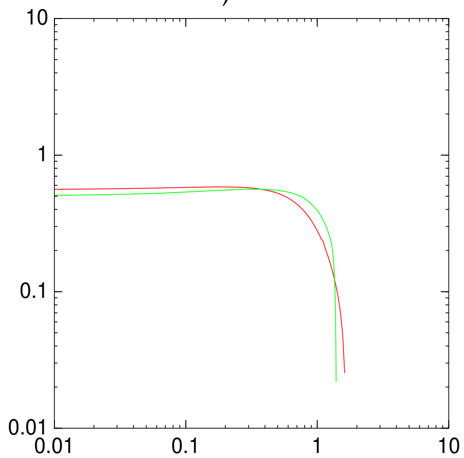


$$d = -31, \ \ B = 2.735, \ \ r = 1.990, \ \ c = 0.643$$

Results 2, negative discriminant

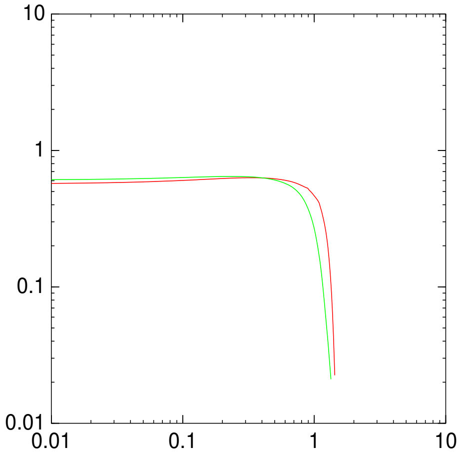


$d = -44, \;\; B = 2.607, \;\; r = 1.648, \;\; c = 0.671$

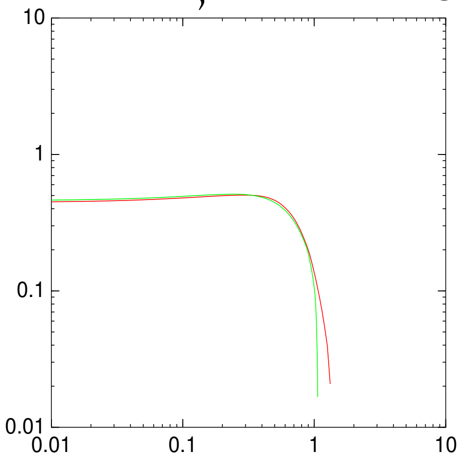


$d = -59, \;\; B = 2.912, \;\; r = 1.699, \;\; c = 1.621$

Results 3, positive discriminant



$d = 49, \;\; B = 2.282, \;\; r = 1.443, \;\; c = 1.032$



$d = 81, \;\; B = 1.839, \;\; r = 1.436, \;\; c = 1.462$

Conclusion

There has been spectacular improvement in the algorithmics of Diophantine approximation in the last few years. The dynamical questions concerning $\mathbb{C}^2$ maps (and related areas: standard map, flows, ...) are still largely open

References

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